

THE NEWTON WONDER IN MECHANICS

*By D. Lynden-Bell**Institute of Astronomy, Cambridge*

Application of Newton's ideas from *Principia* gives many **new** results in mechanics. Here we explore the question "What form of extra force will maintain the **magnitude** of a vector constant of the motion while changing its direction?"

Introduction

Shortly after his death I reviewed Chandrasekhar's last book¹ *Newton's Principia for the Common Reader*. This fine book introduced me to the whole sweep of Newton's ideas in mechanics and sent me back to reading *Principia* itself in translation³. Newton's insight stimulated many new ideas in me of which only the following have been investigated so far.

Revolving orbits

In his theorem on revolving orbits Newton points out that in a central orbit $r^2 d\phi/dt = h$ then $r^2 d(\alpha\phi)/dt = \alpha h$. So an angular motion with $\phi^* = \alpha\phi$ replacing ϕ will still have constant angular momentum. He then asks what extra central force is needed to leave the r motion unchanged and deduces that it must balance the extra centrifugal force $(\alpha^2 - 1)h^2 r^{-3}$. He then deduces that the apses of such orbits rotate at the rate $(\alpha - 1)2\pi/P$ faster. Here P was the mean period in ϕ of the original orbit. The new orbit when viewed from axes revolving at the rate $(\alpha - 1)d\phi/dt$ is exactly the same as the old one viewed from fixed axes. But $(\alpha - 1)d\phi/dt$ is **not** a steady rotation rate. In reference ⁴ we explore the weird distorted shapes taken by such orbits when viewed from axes rotating at the mean rate $(\alpha - 1) \langle d\phi/dt \rangle$ where $\langle d\phi/dt \rangle = 2\pi/P$.

Exact general N-body solutions

Few know that Newton solved completely an N body problem for any N for any initial conditions. The force law for which he did this was

$$\mathbf{F}_{IJ} = km_I m_J (\mathbf{r}_J - \mathbf{r}_I).$$

The total force on body I is then directed at the barycentre at $\bar{\mathbf{r}}$

$$\mathbf{F}_I = \sum \mathbf{F}_{IJ} = km_I M (\bar{\mathbf{r}} - \mathbf{r}_I)$$

where M is the total mass. Thus each body moves as though attracted to the centre of mass which itself travels uniformly. This force law can be found by differentiation of the total potential energy $V = \frac{1}{2}kM^2 r^2$ where

$$r^2 = \sum_{I < J} m_I m_J M^{-2} (\mathbf{r}_J - \mathbf{r}_I)^2.$$

Ruth Lynden-Bell and I have shown⁵ that this result can be generalised to solve the non-harmonic case in which V is replaced by any $V(r)$. Some results even extend to V of the form

$$V_0(r) + r^{-2} V_2 \left(\frac{\mathbf{r}_I - \mathbf{r}_J}{r} \right)$$

where V_0 is an arbitrary function of r and V_2 is any function of its many scale-free arguments. After some struggles with Einstein-Bose & Fermi-Dirac statistics we solved the $V(r)$ problem completely in quantum mechanics⁶. We have further work in progress on the statistical mechanics and mechanics of these extraordinary N-body problems which can be applied to Bose-Einstein condensates.

Vector constants of the motion

We all know that in the presence of a central force the radius vector from the centre to a body sweeps out equal areas in equal times in a plane. However, although Newton proves it, we are not taught the converse that if from some fixed point S , the radius vector to a body sweeps out equal areas in equal times in a plane then the force on

that body is central to S . This is Newton's² proposition 2, but he follows it with a truly remarkable scholium or rider that no-one seems to have understood. If we have a body subjected to a central force but beside that there is another force which is at each moment perpendicular to the current plane of motion, then the radius vector to the body still sweeps out equal areas in equal times, notwithstanding its non-coplanar motion.

To prove this consider the moment of this extra force about the centre. Since the extra force is parallel to the angular momentum the moment is perpendicular to it. Thus the rate of change of the angular momentum is perpendicular to it, so the magnitude or 'length' of the angular momentum vector remains constant. In the surface made up of infinitesimal areas perpendicular to the angular momentum at each instant the rate of sweeping of area $\frac{1}{2}r^2 d\phi/dt = h/2$ is therefore constant QED. This is all so obvious to Newton that he does not stoop to prove it - he simply states it!

Analysing what Newton has done here, he takes a vector constant of the motion and asks under what circumstances the magnitude of that vector can be conserved without requiring each component to be separately conserved. But there are three vector conserved quantities in classical mechanics. They are, besides the angular momentum, the linear momentum \mathbf{p} and $\mathbf{r} - \mathbf{v}t = \mathbf{r}_0$, the position of the barycentre at $t = 0$. Can we preserve $|\mathbf{p}|$ without keeping \mathbf{p} constant? Yes it requires a force $d\mathbf{p}/dt = m\mathbf{v} \times \mathbf{B}$ that is always perpendicular to \mathbf{p} . Thus we are led to forces of gravo-magnetic type of which Coriolis force is a special example. What forces are needed to preserve the magnitude of $\mathbf{r} - \mathbf{v}t$? Evidently $(\mathbf{r} - \mathbf{v}t) \cdot \frac{d}{dt}(\mathbf{r} - \mathbf{v}t) = 0$ so $-\dot{\mathbf{v}}t$ must be of the form $(\mathbf{r} - \mathbf{v}t) \times \mathbf{B}$.

Thus the force per unit mass \mathbf{F} must be of the form

$$\mathbf{F} = \mathbf{v} \times \mathbf{B} - \mathbf{r} \times \mathbf{B}/t \quad (1)$$

where \mathbf{B} may be any function of position, time and velocity. Perhaps the simplest example of such a force occurs when a magnetic field is constant in space but varies its strength with time. The associated electric field is $\mathbf{E} = \frac{1}{2}\mathbf{r} \times d\mathbf{B}/dt$ so the total electromagnetic force on unit charge with unit mass will be of the form (1) provided

$\mathbf{B} = \mathbf{b}t^{-2}$ with \mathbf{b} a constant. We orient the z axis along it so $\mathbf{b} = (0, 0, b)$. Using the fact that $(\mathbf{r} - \mathbf{v}t)^2 = \text{const} = z_0^2 + R_0^2$ the equations of motion may be solved giving

$$\left. \begin{aligned} x + iy &= t(u + we^{ib/t}); & x - v_x t + i(y - v_y t) &= ibwe^{ib/t} \\ z &= v_z t + z_0 \end{aligned} \right\} \quad (2)$$

where u and w are complex constants and v_z & z_0 are real ones. $|w| = R_0/b$. Thus Newton's method of generalising constants of the motion works beautifully for this problem too.

Special relativity

Can the idea be extended to relativity? Here the ten classical integrals combine into the energy-momentum 4-vector, \mathbf{P} and the anti-symmetric tensor

$$K_{ij} = \epsilon_{ijkl} x^k P^l .$$

The invariant scalars are

$$P^2, \quad K_{ij} K^{ij} = 4K^2 \quad \text{and} \quad \epsilon^{ijkl} K_{ij} K_{kl} .$$

Unfortunately $P^2 = m_0^2 c^2$ which is always constant and the epsilon KK invariant is zero identically, so only K^2 is suitable for playing Newton's game. Differentiating with respect to comoving time τ and remembering that $m d\mathbf{x}/d\tau = \mathbf{P}$, we find that K^2 is constant when the four-force $\mathbf{F} = d\mathbf{P}/d\tau$ is perpendicular to $\mathbf{x} = (ct, x, y, z)$. However, since P^2 is constant the four-force is always perpendicular to \mathbf{P} . If \mathbf{N} is any vector along a third independent direction perpendicular to the four-force, that force takes the form $F_i = \epsilon_{ijkl} x^j P^k N^l$ so this is the general form of force when $K^2 = (\mathbf{r} \times \mathbf{p})^2 - E^2 c^{-2} (\mathbf{r} - \mathbf{v}t)^2$ is conserved. Here E is the energy. A pretty example is given by taking \mathbf{N} along the time direction; then reverting to 3-vectors and writing $N = |\mathbf{N}|$

$$d\mathbf{p}/d\tau = N \mathbf{r} \times \mathbf{p} . \quad (3)$$

Under such forces $|\mathbf{p}|$ is clearly constant so $|\mathbf{v}| = v$ is constant too; furthermore writing $\mathbf{L} = \mathbf{r} \times \mathbf{p}$ we see that $d\mathbf{L}/d\tau = N \mathbf{r} \times \mathbf{L}$ so $|\mathbf{L}|$ is constant. From our K^2 invariant this

of course implies $|\mathbf{r} - \mathbf{v}t|$ is constant. \mathbf{p} and \mathbf{L} are perpendicular and obey the same equation (3) which corresponds to angular rotation at the variable rate $N\mathbf{r}\sqrt{1 - v^2/c^2}$. A third vector perpendicular to the other two is given by $\mathbf{R} = \mathbf{p} \times \mathbf{L}/p^2$ which is the component of \mathbf{r} transverse to \mathbf{p} . It is readily shown that it too obeys $d\mathbf{R}/d\tau = N\mathbf{r} \times \mathbf{R}$, so $|\mathbf{R}|$ is constant and $\mathbf{R}, \mathbf{p}, \mathbf{L}$ form a triply orthogonal triad of rotating vectors of constant length. \mathbf{v} is along \mathbf{p} and is also fixed in that rotating frame. Now

$$d(\mathbf{r} \cdot \mathbf{v})/dt = v^2 = \text{const},$$

so

$$\mathbf{r} = \mathbf{R} + \mathbf{v}(t + \alpha) \quad (4)$$

since the components of \mathbf{r} transverse to \mathbf{v} make up \mathbf{R} . The above equation gives the misleading impression that the particle is travelling uniformly in a straight line. That is indeed true **relative to the rotating axes** but in absolute axes \mathbf{R} and \mathbf{v} are rotating and at a time dependent rate. Notice that $\mathbf{X} = \mathbf{r} - \mathbf{v}t$ is constant in the rotating axes so \mathbf{X} moves on a sphere in fixed axes i.e., $|\mathbf{X}| = \text{const}$. Viewed from the rotating axes in which \mathbf{R}, \mathbf{p} and \mathbf{L} are fixed constant vectors the whole motion is simple and in those axes even their rotation rate takes the simple form,

$$\boldsymbol{\Omega} = (1 - v^2/c^2)^{\frac{1}{2}} N \mathbf{r} = (1 - v^2/c^2)^{\frac{1}{2}} N [\mathbf{R} + \mathbf{v}(t + \alpha)] \quad (5)$$

although it varies in magnitude and direction. Of course from the viewpoint of a co-rotating observer the universe rotates at rate $-\boldsymbol{\Omega}$. The true complication of the motion is only comprehended when one imagines what is seen by an inertial observer to whom the inclined offset straight line, down which the motion occurs uniformly, whirls around a changing axis at a non-uniform rate. So far the reader may have assumed that N was a constant but nowhere have we assumed this and indeed the force law can be such that N is any scalar function of \mathbf{r}, \mathbf{v} and t .

In parting we notice that this is but a very special case of the force laws that preserve K^2 . We chose it because the others necessarily involve t explicitly but it is peculiar in

that \mathbf{N} 's direction is fixed (along the time direction) and not varying as it would in the general case. (3) is in fact a special case of (1) with \mathbf{B} radial.

The eccentricity vector and magnetic monopoles

To return to where Newton started, his equation of motion for a particle moving under a central force $-V'\hat{\mathbf{r}}$ with an extra force $N\mathbf{r} \times \mathbf{v}$ reads for unit mass

$$d\mathbf{v}/dt = -V'\hat{\mathbf{r}} + N\mathbf{r} \times \mathbf{v} \quad (6)$$

with the central force present can we still find some rotating axes in which the motion is especially simple? Letting dots denote rates of change with respect to the moving axes: $d\mathbf{v}/dt = \dot{\mathbf{v}} + \boldsymbol{\Omega} \times \mathbf{v}$. Evidently we should again take $\boldsymbol{\Omega} = N\mathbf{r}$. Then $d\mathbf{r}/dt = \dot{\mathbf{r}} = \mathbf{v}$ so even when N varies

$$\ddot{\mathbf{r}} = -V'\hat{\mathbf{r}} \quad (7)$$

so if we take the rotating axes the whole dynamics is the same as if the N force were absent and we had fixed axes. Thus in the rotating axes we have $\mathbf{r} \times \dot{\mathbf{r}} = \mathbf{h} =$ constant relative to the rotating axes and when $V = -GM/r$ we write $\hat{\mathbf{r}} = \mathbf{r}/r$ and obtain

$$\left. \begin{aligned} \mathbf{h} \times \ddot{\mathbf{r}} &= -GM \left(\hat{\mathbf{r}} \times \frac{\dot{\mathbf{r}}}{r} \right) \times \hat{\mathbf{r}} = -GM\dot{\hat{\mathbf{r}}} \\ \text{so} \quad \mathbf{h} \times \dot{\mathbf{r}} &= -GM(\hat{\mathbf{r}} + \mathbf{e}) \end{aligned} \right\} \quad (8)$$

where \mathbf{e} is a vector constant of integration. \mathbf{e} is constant in the rotating axes not the fixed ones. Those with no history call it the Runge Lenz Vector¹ but to me it is Hamilton's eccentricity vector pointing to pericentre with magnitude equal to the eccentricity. Evidently

$$\mathbf{e} = \dot{\mathbf{r}} \times \mathbf{h}/(GM) - \hat{\mathbf{r}} = \mathbf{v} \times \mathbf{h}/(GM) - \hat{\mathbf{r}} \quad (9)$$

¹Hamilton⁷ gave the first coordinate free derivation in 1845 by writing the equations in quaternion form. Bernoulli⁸, Laplace⁹ and possibly even Newton got the result earlier. Atomic Physicists learned the result from Runge & Lenz much later. See Goldstein¹⁰

again \mathbf{e} is constant in our rotating axes. \mathbf{h} is **not** constant in the fixed axes but obeys $d\mathbf{h}/dt = N\mathbf{r} \times \mathbf{h}$.

We are now in a position to ask a question that crossed my mind but which I would not have remembered to investigate had not S Aarseth reminded me. What is the most general force law that leaves $|\mathbf{e}|$ invariant while letting \mathbf{e} change?

Evidently the example given above has this property since \mathbf{e} precesses with the axes at the rate $\boldsymbol{\Omega} = N\mathbf{r}$ which may well be variable in magnitude as well as direction; however that example does not give the general case. We now revert to fixed inertial axes. Again $\mathbf{h} = \mathbf{r} \times \mathbf{v}$ whether that quantity is constant or not and \mathbf{e} is defined by (9) with \mathbf{v} written for $\dot{\mathbf{r}}$.

Differentiating $GM\mathbf{e}$ we have from (9)

$$GM \frac{d\mathbf{e}}{dt} = \frac{d\mathbf{v}}{dt} \times \mathbf{h} + \mathbf{v} \times \left(\mathbf{r} \times \frac{d\mathbf{v}}{dt} \right) - GM \frac{d\hat{\mathbf{r}}}{dt} .$$

The extra force per unit mass that causes the change in \mathbf{e} must be $\mathbf{F} = \frac{d\mathbf{v}}{dt} + GMr^{-2}\hat{\mathbf{r}}$, so

$$GM \frac{d\mathbf{e}}{dt} = \mathbf{F} \times \mathbf{h} + \mathbf{v} \times (\mathbf{r} \times \mathbf{F}) \quad (10)$$

We want this extra force to leave e^2 unchanged so there must be no component of $d\mathbf{e}/dt$ along \mathbf{e} . This condition may be put in the form

$$\mathbf{F} \cdot \boldsymbol{\ell} = 0 \quad (11)$$

where

$$\boldsymbol{\ell} = \mathbf{h} \times \mathbf{e} + (\mathbf{e} \times \mathbf{v}) \times \mathbf{r} . \quad (12)$$

Thus the general \mathbf{F} that leaves e^2 invariant is of the form

$$\mathbf{F} = \mathbf{k} \times \boldsymbol{\ell} \quad (13)$$

and it will cause a change of \mathbf{e} given by

$$GM \, d\mathbf{e}/dt = (\mathbf{k} \times \boldsymbol{\ell}) \times \mathbf{h} + \mathbf{v} \times (\mathbf{r} \times (\mathbf{k} \times \boldsymbol{\ell})) \quad (14)$$

since the last term of $\boldsymbol{\ell}$ itself involves two cross products we see that the final term above involves five consecutive cross products and it takes some work to sort them out and to re-express the RHS as a cross product with \mathbf{e} so that

$$d\mathbf{e}/dt = \boldsymbol{\omega} \times \mathbf{e} \quad (15)$$

We write

$$\mathbf{C} = (\mathbf{k} \cdot \mathbf{r})\mathbf{e} + (\mathbf{k} \cdot \mathbf{e})\mathbf{r}$$

and notice it is in the plane of motion and find

$$GM\boldsymbol{\omega} = 2(\mathbf{k} \cdot \mathbf{h})\mathbf{h} + (\mathbf{v} \cdot \mathbf{r}) (\mathbf{C} \times \mathbf{v}) \times \mathbf{e}/e^2 \quad (16)$$

Thus our result is that the general force is of the form (13) in which \mathbf{k} is any vector (which may vary with time) while it leaves e^2 invariant. The angular velocity of \mathbf{e} is given by (16). The first term causes \mathbf{e} to swing in the plane of motion while the second swings it down through that plane ($\mathbf{C} \times \mathbf{v}$ is parallel to \mathbf{h}). A useful final check on the complicated derivation is that if any multiple of $\boldsymbol{\ell}$ is added to \mathbf{k} it must make no difference. Indeed $\mathbf{C} \times \mathbf{v}$ changes by $(\boldsymbol{\ell} \cdot \mathbf{r} \mathbf{e} + \boldsymbol{\ell} \cdot \mathbf{e} \mathbf{r}) \times \mathbf{v}$ which is identically zero by (12).

The general case discussed above requires complicated forces of the form given by (12) and (13) but a great simplification occurs if we demand that these forces do no work. The condition $\mathbf{F} \cdot \mathbf{v} = 0$ leads *via* (12) & (13) to $\mathbf{k} \cdot \mathbf{h} = 0$ and since we can always take \mathbf{k} perpendicular to $\boldsymbol{\ell}$ we set $\mathbf{k} = N\boldsymbol{\ell} \times \mathbf{h}/\ell^2$. Then the extra force \mathbf{F} reverts to Newton's simple form, $N\mathbf{r} \times \mathbf{v}$ discussed earlier (6). We may consider this as a magnetic force by endowing our particle with unit charge and writing $\mathbf{B} = -cN\mathbf{r}$. It is then natural to require that $\text{div}\mathbf{B} = 0$ so $r^{-2}d/dr(Nr^3) = 0$ and therefore $cNr^3 = -Q = \text{constant}$. The resulting field is $\mathbf{B} = Q\hat{\mathbf{r}}/r^2$ which is clearly the field of a magnetic monopole of strength Q . As Poincaré¹¹ first found, motion in such fields has a singular beauty. For a particle of mass m and charge q orbiting a monopole of charge $-Zq$ and monopole strength Q

$$m\ddot{\mathbf{r}} = -\zeta\hat{\mathbf{r}}r^{-2} + \eta\hat{\mathbf{r}} \times \mathbf{v}r^{-2}$$

where $\zeta = Zq^2$ and $\eta = Qq/c$. Cross multiplying by \mathbf{r} , $d\mathbf{L}/dt = -\eta\mathbf{h} \times \hat{\mathbf{r}}/r^2 = -\eta(\hat{\mathbf{r}} \times \mathbf{v}) \times \hat{\mathbf{r}}/r = -\eta d\hat{\mathbf{r}}/dt$ where $\mathbf{L} = m\mathbf{h}$. It follows that L^2 is constant and that

$$\mathbf{L} - \eta\mathbf{r} = \mathbf{J} = \text{const.}$$

From which it follows that $\mathbf{L}^2 = \mathbf{J} \cdot \mathbf{L}$ and $\eta = -\mathbf{J} \cdot \hat{\mathbf{r}}$ so \mathbf{L} and \mathbf{r} precess around \mathbf{J} on coaxial cones. The orbit is an ellipse that precesses around the \mathbf{r} cone because of the deficit angle that is missing when the cone is flattened, but I gave the details earlier and solved the quantum mechanics of monopolar hydrogen. The results here can be extended to give the interesting motions of a small spinning sphere both classically and in quantum mechanics ¹². Even the Dirac equation has a pretty solution in the field of a charged monopole. In general relativity the equivalent of a monopole is a gravomagnetic monopole and the required generalisation of Schwarzschild's metric is called, NUT space, it has many weird properties. Although the space has the same properties when viewed from any direction its metric is not spherically symmetric and can not be transformed into spherically symmetric form ¹². Thus following a lead from Newton one ends up challenging the concept of spherical symmetry in General Relativity!

Nouri-Zonoz and I have explored the gravitational lensing of gravomagnetic monopoles ¹³ and he has found the cylindrically symmetrical space corresponding to a line gravomagnetic monopole ¹⁴ - the metric can not be put into a single-valued cylindrically symmetric form!

What we have found so far comes from a tiny fraction of *Principia*; we hope that it will encourage others to read Chandrasekhar's last book and Newton's great one!

References

- (1) S. Chandrasekhar, *Newton's Principia for the Common Reader* (Oxford University Press), 1995.
- (2) I. Newton, *Principia* (R.Soc., London), 1687.
- (3) F. Cajori, *Newton's Principia, (Motte's Translation Revised)* (University of California Press, Berkeley), 1934.
- (4) D. Lynden-Bell & R.M. Lynden-Bell, *R. Soc. Notes. & Record*, **51**, 195.
- (5) D. Lynden-Bell & R.M. Lynden-Bell, *Proc. R. Soc. A.*, **455**, 475, 1999.
- (6) D. Lynden-Bell & R.M. Lynden-Bell, *Exact Quantum Solutions to Extraordinary N-body Problems*, *Proc. R. Soc. A.*, accepted).
- (7) W. R. Hamilton, *Proc. R. Irish Acad.*, **III**, Appendix p.36, 1845.
- (8) J. I. Bernoulli, *Hist. Academie Royale*, p. 523, 1712.
- (9) P. S. Laplace, *Traité de mécanique celeste (Vol 1)* (Paris Chez J.B.M. Duprat), 1799.
- (10) H. Goldstein, *Am. J. Phys.*, **44**, 1123, 1976.
- (11) H. Poincaré, *CR Acad. Sci.*, **123**, 530, 1896.
- (12) D. Lynden-Bell, & M. Nouri-Zonoz, *Revs. Mod. Phys.*, **70**, 427, 1998.

(13) M. Nouri-Zonoz & D. Lynden-Bell, *MNRAS*, **292**, 714, 1997.

(14) M. Nouri-Zonoz, *Class. Quantum Grav.*, **14**, 3123, 1997.